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# A new class of solvable many-body problems with constraints, associated with an exceptional polynomial subspace of codimension 2

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## Abstract

A new class of many-body models is identified and investigated. Just as those we recently discovered, these many-body problems are *solvable* provided the initial data satisfy certain constraints; for such data the solution of the initial-value problem can be achieved via *algebraic* operations, such as finding the zeros of known polynomials. *Entirely isochronous* subclasses of these models are also exhibited, whose *generic* solutions are *completely periodic* with the *same* fixed period in their *entire* phase space.

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## 1. Introduction and main results

In a previous paper [4], two novel classes of *solvable* many-body problems of *goldfish* type were identified and investigated. We refer to it for a more detailed description than provided herein of the general strategy to identify these *solvable* systems, including the definition of our terminology (solvable, goldfish, exceptional polynomials subspaces and so on). In the present paper new models of this kind are studied. They are obtained by considering the motion of the $N$ roots $z_n(t)$ of an (appropriately defined, time-dependent, monic) polynomial $\psi(z, t)$, of degree $N$ in $z$, belonging to the *exceptional polynomial subspace* $X_2$ of the polynomials in $z$ of degree $m$ up to $N$, whose first derivative vanishes at two fixed points, say at $z = \pm 1$. A convenient basis in this space reads as follows [5]:

$$X_2 = \text{span}\{\pi_0(z), \pi_3(z), \pi_4(z), \ldots, \pi_N(z)\}, \tag{1a}$$

$$\pi_m(z) = z^m - \varepsilon_m \frac{m}{2} z^2 - \varepsilon_{m+1} m z, \qquad m = 0, 1, \ldots, N, \tag{1b}$$

where (here and hereafter)

$$\varepsilon_m = \begin{cases} 1, & m \text{ even} \\ 0, & m \text{ odd.} \end{cases} \tag{2}$$

This polynomial subspace has a codimension two (with respect to the space of the polynomials of degree up to $N$); indeed this definition of the basic polynomials $\pi_m(z)$ implies the identities $\pi_1(z) \equiv \pi_2(z) \equiv 0$, in addition to $\pi_0(z) = 1$, entailing for the (time-dependent, monic) polynomial $\psi(z, t)$ the representation

$$\psi(z, t) = \pi_N(z) + \sum_{m=1}^{N-3} [c_m(t) \pi_{N-m}(z)] + c_N(t) \tag{3a}$$

as well as the constraint

$$\psi_z(\pm 1, t) = 0. \tag{3b}$$

Here and hereafter $N$ is a *positive integer* larger than 2, $N \geqslant 3$, and the subscripted variables denote partial differentiations.

To identify a class of *solvable $N$-body problems*, we now focus on the time evolution of the $N$ zeros $z_n(t)$ of this polynomial $\psi(z, t)$,

$$\psi(z, t) = \prod_{n=1}^{N} [z - z_n(t)], \tag{3c}$$

interpreting them as the coordinates of $N$ particles, generally moving in the complex $z$-plane; this interpretation is justified by the Newtonian (acceleration equal force) character of the corresponding equations of motion, see below, which obtain from the assumption that $\psi(z, t)$ evolve in time according to a *linear*, *second-order*, *constant-coefficient* partial differential equation (PDE), implying that the corresponding time evolution of the $N - 2$ coefficients $c_m(t)$, see (3a), is determined by a system of *linear* ordinary differential equations (ODEs) with *constant* coefficients (see below); hence it is *solvable* by purely algebraic techniques.

To identify this PDE we take advantage of the characteristic property of the space $X_2$, to be preserved under the action of the following three *linear* differential operators [5]:

$$T_0 = \left( \frac{d}{dz} \right)^2 + z \frac{z^2 - 5}{z^2 - 1} \left( \frac{d}{dz} \right) - N, \tag{4a}$$

$$T_1 = z \left( \frac{d}{dz} \right)^2 - 2 \frac{z^2 + 1}{z^2 - 1} \left( \frac{d}{dz} \right), \tag{4b}$$

$$T_2 = (z^2 - 1) \left( \frac{d}{dz} \right)^2 - 2z \left( \frac{d}{dz} \right) - N(N - 3). \tag{4c}$$

(Note that although $T_0$ and $T_1$ have *rational* coefficients, they do possess an infinite number of *polynomial* eigenfunctions [5, 6].) Indeed, as can be easily verified, the action of these operators on the basic elements $\pi_m(z)$ reads as follows:

$$T_0[\pi_m] = (m - N)\pi_m + m(m - 5)\pi_{m-2} - 4m \sum_{\ell=4}^{m-3} [\varepsilon_\ell \pi_{m-\ell}] - m \varepsilon_m \pi_0, \tag{5a}$$

$$T_1[\pi_m] = m(m - 3)\pi_{m-1} - 4m \sum_{\ell=2}^{m-4} [\varepsilon_\ell \pi_{m-1-\ell}] - 2m \varepsilon_{m+1} \pi_0, \tag{5b}$$

$$T_2[\pi_m] = [m(m-3) - N(N-3)]\pi_m - m(m-1)\pi_{m-2} + m\varepsilon_m\pi_0. \tag{5c}$$

Here and hereafter, we adopt the usual convention according to which a sum vanishes if its lower limit exceeds its upper limit.

Accordingly, the most general *linear*, *constant-coefficient*, *second-order* PDE maintaining $\psi(z,t)$ within $X_2$, namely being consistent with the representation ($3a$) with ($3a$), reads as follows:

$$\psi_{tt} + a_1\psi_t + (a_2 T_0 + a_3 T_1 + a_4 T_2)\psi = 0 \tag{6a}$$

or equivalently, via (4),

$$\psi_{tt} + a_1\psi_t + [a_2 + a_3 z + a_4(z^2 - 1)]\psi_{zz} + \left[(a_2 - 2a_4)z - 2a_3 - 4\frac{a_2 z + a_3}{z^2 - 1}\right]\psi_z$$
$$- [Na_2 + N(N-3)a_4]\psi = 0. \tag{6b}$$

Here and hereafter, we assume that the four parameters $a_j$ are *a priori* arbitrary (possibly *complex*) constants.

Via (5), through a completely straightforward if tedious computation, this PDE entails that the $N-2$ coefficients, $c_m(t)$, $m = 1, \ldots, N-3$, and $c_N(t)$, satisfy the following system of $N-2$ ODEs:

$$\ddot{c}_m + a_1\dot{c}_m - m[a_2 + (2N - m - 3)a_4]c_m + (N - m + 1)(N - m - 2)a_3 c_{m-1}$$
$$+ (N - m + 2)[(N - m - 3)a_2 - (N - m + 1)a_4]c_{m-2}$$
$$- 4(N - m + 3)a_3 c_{m-3} - 4\sum_{\ell=0}^{m-4}[(N - \ell)(\varepsilon_{m-\ell}a_2 + \varepsilon_{m-\ell+1}a_3)c_\ell] = 0,$$
$$m = 1, \ldots, N-3, \tag{7a}$$
$$\ddot{c}_N + a_1\dot{c}_N - N[a_2 + (N-3)a_4]c_N - 6a_3 c_{N-3}$$
$$- \sum_{\ell=0}^{N-4}\{(N-\ell)[\varepsilon_{N-\ell}(a_2 - a_4) + 2\varepsilon_{N-\ell+1}a_3]c_\ell\} = 0. \tag{7b}$$

These equations are written on the understanding (see ($3a$)) that $c_0(t) = 1$ and $c_m(t) = 0$ if $m < 0$.

Let us re-emphasize that the solution of this system, (7), of $N-2$ *linear* ODEs with *constant* coefficients is a purely *algebraic* task (see below).

The corresponding time evolution of the $N$ zeros $z_n(t)$ of the polynomial $\psi(z,t)$ constitutes therefore a *solvable* dynamical system, interpretable as an $N$-body problem when one identifies $z_n(t)$ as the coordinate at time $t$ of the $n$th (point) particle, generally moving in the complex $z$-plane. This interpretation is now justified by the observation (proven in section 2) that the factorized representation ($3c$) of the monic polynomial $\psi(z,t)$ via its zeros, when inserted in the linear PDE ($6b$) satisfied by $\psi(z,t)$, entails that the $N$ zeros $z_n(t)$ evolve according to the Newtonian equations of motion

$$\ddot{z}_n = -a_1\dot{z}_n + a_2 z_n \frac{z_n^2 - 5}{z_n^2 - 1} - 2a_3 \frac{z_n^2 + 1}{z_n^2 - 1} - 2a_4 z_n$$
$$+ 2\sum_{m=1,m\neq n}^{N} \frac{\dot{z}_n\dot{z}_m + a_2 + a_3 z_n + a_4(z_n^2 - 1)}{z_n - z_m}, \qquad n = 1, \ldots, N. \tag{8a}$$

But of course the *solvable* character of this $N$-body problem hinges on the validity of the representation (3a), entailing restriction (3b) that (as shown in section 2) corresponds to the following four conditions on the *initial* data of this $N$-body problem:

$$\sum_{n=1}^{N} \frac{1}{z_n(0) \pm 1} = 0, \qquad \sum_{n=1}^{N} \frac{\dot{z}_n(0)}{[z_n(0) \pm 1]^2} = 0. \qquad (8b)$$

The novelty of this *solvable* $N$-body problem is of course due to the *rational* dependence on the coordinates $z_n$ featured by the second and third terms on the right-hand side of the equations of motion (8a), associated with the 'coupling constants' $a_2$ and $a_3$ and originating from the operators $T_0$ and $T_1$, see (4).

### 1.1. Behaviour of the solvable N-body problem (8)

The *general* solution of the system of $N - 2$ *linear* evolution ODEs (7) satisfied by the $N - 2$ coefficients $c_m(t)$, $m = 1, \ldots, N - 3$, and $c_N(t)$ reads

$$c_m(t) = \sum_{\ell=1; l \neq N-1, N-2}^{N} \left\{ \gamma^{(\ell,+)} u_m^{(\ell,+)} \exp[\lambda^{(\ell,+)} t] + \gamma^{(\ell,-)} u_m^{(\ell,-)} \exp[\lambda^{(\ell,-)} t] \right\},$$
$$m = 1, \ldots, N - 3, N, \qquad (9)$$

where the two $(N - 2)$ coefficients $\gamma^{(\ell,\pm)}$ are *a priori arbitrary* (to be fixed by the *initial* data). The quantities $\lambda^{(\ell,\pm)}$ and $u_m^{(\ell,\pm)}$ are the eigenvalues and the (components of the), corresponding eigenvectors of the generalized (algebraic) eigenvalue problem

$$\sum_{m=1; m \neq N-2, N-1}^{N} M_{nm} u_m^{(\ell,\pm)} = \lambda^{(\ell,\pm)} (\lambda^{(\ell,\pm)} + a_1) u_n^{(\ell,\pm)}, \qquad \ell = 1, \ldots, N - 3, N, \qquad (10a)$$

where the definition of the $(N - 2) \times (N - 2)$ matrix $M$ is evident from (7). The *triangular* character of this matrix $M$ (see (7)) entails that the two $(N - 2)$ eigenvalues $\lambda^{(\ell,\pm)}$ are the two $(N - 2)$ solutions of the $(N - 2)$ *decoupled* second-degree equations

$$\lambda^2 + a_1 \lambda - \ell[a_2 + (2N - \ell - 3)a_4] = 0, \qquad \ell = 1, \ldots, N - 3, N. \qquad (10b)$$

Hence

$$\lambda^{(\ell,\pm)} = \frac{-a_1 \pm \Delta_\ell}{2}, \qquad \Delta_\ell^2 = a_1^2 + 4\ell[a_2 + (2N - \ell - 3)a_4],$$
$$\ell = 1, \ldots, N - 3, N. \qquad (10c)$$

These findings entail that, for generic (possibly *complex*) values of the 'coupling constants' $a_j$, the asymptotic behaviour at large time of the solutions of the $N$-body problem (8) can be inferred from the treatment provided in appendix G of [2] (entitled 'asymptotic behaviour of the zeros of a polynomial whose coefficients diverge exponentially'). In the special case when $\Delta_\ell$ vanishes for some relevant value of $\ell$, the asymptotic behaviour at large time of the polynomial $\psi(z, t)$ (see (3a) with (1) and (9)) generally also contains a term linear in $t$—which might become dominant as $t \to \infty$ if all the other terms vanish exponentially in this limit. While clearly, if $a_1$ and (for all the relevant values of $\ell$) $\Delta_\ell$ are *imaginary*, then *all* motions of the $N$-body problem (8) remain *confined* for all time. And finally (most interestingly) if

$$a_1 = ik_1\omega, \qquad a_2 = k_4[(2N - 3)k_4 - k_1]\omega^2, \qquad a_4 = k_4^2\omega^2, \qquad (11a)$$

yielding

$$\lambda^{(\ell,+)} = i\ell k_4\omega, \qquad \lambda^{(\ell,-)} = -i(\ell k_4 + k_1)\omega, \qquad \ell = 1, \ldots, N - 3, N, \qquad (11b)$$

with $\omega$ an arbitrary *positive* constant (having the significance of a circular frequency) and $k_1$, $k_4$ the two *arbitrary integers* (*positive* or *negative* but *not vanishing*, to avoid that the eigenvalue $\lambda$ vanish, and respecting the restriction $k_1 \neq -2\ell k_4$ for $\ell = 1, \ldots, N-3, N$, required to guarantee that $\lambda^{(\ell,+)} \neq \lambda^{(\ell,-)}$), then clearly the $N$-body problem (8) is *entirely isochronous*; its *generic* (*complex!*) solutions—in its *entire* phase space, except possibly for a subregion of vanishing measure characterized by the solutions that become *singular* due to the collision of two or more particles—are *completely periodic*,

$$z_n(t + \tilde{T}) = z_n(t), \qquad n = 1, 2, \ldots, N, \tag{11c}$$

with a common period $\tilde{T}$ (possibly not the primitive period in all sectors of phase space) which is a (generally small [7, 3]) *integer* multiple of a basic period, itself a *rational* multiple (depending in an obvious manner on the *integers* $k_1$ and $k_4$) of the standard period $T = 2\pi/\omega$ associated with the circular frequency $\omega$.

Clearly this discussion entails that conditions (11a) are *sufficient* to guarantee that the $N$-body problem (8), with $N$ arbitrary ($N \geqslant 3$), is *entirely isochronous*; for $N > 3$ (but not for $N = 3$, see below) they are also *necessary*.

It is remarkable that the *qualitative* behaviour of this $N$-body problem, (8), turns out to be quite independent of the value of the coupling constant $a_3$, but this fact is obvious when looking at the system of linear evolution equations (7) for the coefficients, since the eigenvalues of the triangular matrix $M$ do not depend on the parameter $a_3$ (but the eigenvectors of course do).

### 1.2. Example: the three-body case

For $N = 3$, system (7) reduces to the single ODE

$$\ddot{c}_3 + a_1 \dot{c}_3 - 3a_2 c_3 = 6a_3, \tag{12a}$$

whose *general* solution reads (provided $a_2 \neq 0$, as we hereafter assume)

$$c_3(t) = \frac{2a_3}{a_2} + \gamma_+ \exp(\lambda_+ t) + \gamma_- \exp(\lambda_- t), \tag{12b}$$

where $\gamma_\pm$ are the two arbitrary constants and $\lambda_\pm$ are the two roots of the equation

$$\lambda^2 + a_1 \lambda - 3a_2 = 0; \tag{12c}$$

hence

$$\lambda_\pm = \tfrac{1}{pwd2}\left[-a_1 \pm \left(a_1^2 + 12a_2\right)^{1/2}\right]. \tag{12d}$$

The particle positions $z_n(t)$ evolving according to the three-body problem (8) are the three roots of the cubic equation in $z$,

$$z^3 - 3z + c_3(t) = 0, \tag{13a}$$

as implied by (3a), (3c) and (1b); hence they satisfy the restrictions

$$z_1(t) + z_2(t) + z_3(t) = 0, \tag{13b}$$

$$z_1(t)z_2(t) + z_2(t)z_3(t) + z_3(t)z_1(t) = -3, \tag{13c}$$

and their time evolution is determined (by these two formulae and) by the rule

$$z_1(t)z_2(t)z_3(t) = -c_3(t), \tag{13d}$$

with $c_3(t)$ given by (12b) where the two parameters $\gamma_\pm$ are determined by the *initial* position and velocity of one of the three particles (those of the other two are then fixed by the four
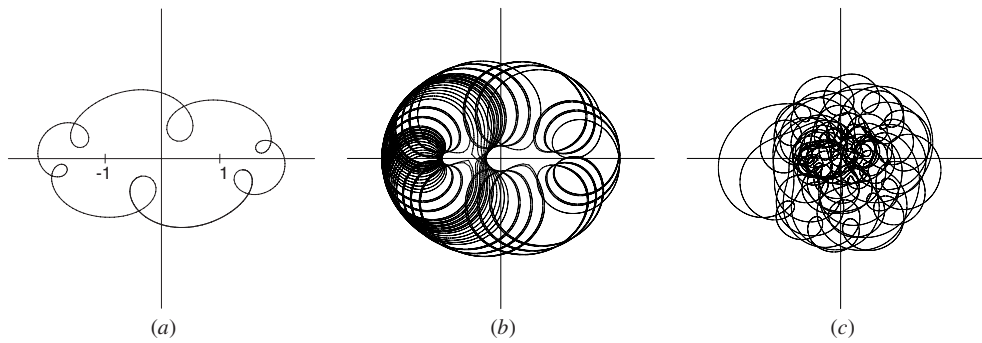
**Figure 1.** Trajectories in the complex $z$-plane of one particle of the three-body system ($8a$) in the periodic case ($a$), and in the quasi-periodic and chaotic cases ($b$) and ($c$) up to $t = 30$.

restrictions ($8b$), or equivalently by ($13b$) and ($13c$), and their time derivatives evaluated at the initial time $t = 0$).

The fact that the solutions of the three-body problem (8) with $N = 3$ are compatible with these formulae is not quite obvious, but it is of course implied by our treatment. This entails that the behaviour of this model does not depend at all on the coupling constant $a_4$; in fact it can be shown via ($13b$) and ($13c$)—or equivalently via the two equations (15) with $N = 3$—that in this $N = 3$ case the terms proportional to $a_4$ on the right-hand side of ($8a$) cancel exactly. Moreover, the dependence of the solutions on the coupling constant $a_3$ is rather trivial (see ($12b$)) and certainly does not affect the *qualitative* behaviour of the system. The values of the other two coupling constants, $a_1$ and $a_2$, are instead significant (see ($12d$)); in particular this model is *entirely isochronous* provided

$$a_1 = 2\mathrm{i}j\omega, \qquad a_2 = (j^2 - k^2)\omega^2/3 \quad \text{so that} \quad \lambda_\pm = \mathrm{i}(j \pm k)\omega, \qquad (14)$$

with $\omega$ an arbitrary *positive* constant and $j, k$ the two *integers* (*arbitrary* except for the two restrictions $k \neq 0$ so that $\lambda_+ \neq \lambda_-$, and $k^2 \neq j^2$ so that $a_2 \neq 0$). This includes the special case with $a_1$ *vanishing* and $a_2$ an arbitrary *negative* constant, when the equations of motion ($8a$) are *real* (if also $a_3$ is *real*). But, especially when the equations of motion are *real*, the *complex* initial data must generally be assigned in order to avoid that the motion run into a singularity due to a particle collision.

Some representative trajectories of one of the three coordinates $z_n(t)$ in the *complex* $z$-plane are displayed in figure 1.

(a) The first trajectory shows a *periodic* solution which corresponds to the initial data that satisfy constraints ($8b$) and to the coupling constants that satisfy the *isochronicity* condition (14), specifically $a_1 = 8\pi\mathrm{i}$, $a_2 = 4\pi^2$, $a_3 = a_4 = 0$, corresponding to $\omega = 2\pi$, $j = 2$ and $k = 1$. In this case all the three particles follow this trajectory with a period $T = 3$.

(b) The second trajectory shows a *quasi-periodic* solution, which corresponds to the same initial data as in the previous case, but to the following values of the coupling constants: $a_1 = 8\pi\mathrm{i}$, $a_2 = 39.2139$, $a_3 = a_4 = 0$ corresponding (see (14)) to $\omega = 2\pi$, $j = 2$ and $k = 1.01$.

(c) The third trajectory shows a *chaotic* solution which corresponds to the same coupling constants as in case (a), but to the initial data which do *not* satisfy constraint ($8b$).

## 2. Proofs

In this section, we show how to arrive at the results reported above whose proofs are not quite obvious from the previous treatment.

Let us begin by proving that the condition (3b) satisfied by the polynomial $\psi(z, t)$ entails via (3c), for the zeros $z_n(t)$, the conditions

$$\sum_{n=1}^{N} \frac{1}{z_n(t) \pm 1} = 0, \tag{15}$$

which are then of course automatically consistent with the equations of motion (8a) and therefore hold for all time provided they hold initially together with their time derivatives, see (8b).

Indeed the factorized representation (3a) entails (by logarithmic differentiation)

$$\psi_z(z, t) = \psi(z, t) \sum_{n=1}^{N} [z - z_n(t)]^{-1}; \tag{16}$$

hence, together with (3b), it implies (15).                                                                               □

Next let us indicate how to prove that the PDE (6b) yields, via (3c) with (15), the equations of motion (8a). This is a standard task, easily accomplished by using the formulae—implied by (3c) and analogous to (16)—easily obtained and in any case available in the literature (see, in particular, section 2.3.2 of [2], or the analogous but more complete lists displayed in [1, 3]), as well as the following two neat formulae (also easily obtained by standard techniques, but using now *both* (3c) *and* (15)):

$$\frac{1}{z^2 - 1} \psi_z = \psi \sum_{n=1}^{N} (z - z_n)^{-1} \frac{1}{z_n^2 - 1}, \qquad \frac{z}{z^2 - 1} \psi_z = \psi \sum_{n=1}^{N} (z - z_n)^{-1} \frac{z_n}{z_n^2 - 1}. \tag{17}$$

## 3. Summary and outlook

In this paper, we identified a new class of solvable many-body problems 'of goldfish type', which are solvable by purely algebraic operations provided the positions of the moving particles satisfy two overall constraints, which are of course compatible with the time evolution and therefore need to be imposed, in the context of the initial-value problem, only on the initial data.

This class of many-body problems contains four arbitrary 'coupling constants'; sufficient restrictions on their values are also identified guaranteeing that the generic motions are completely periodic with the same period (isochronous models). A few representative instances of motions featured by $N$-body problems of this kind (with $N = 3$—both in the isochronous and non-isochronous cases—have been exhibited.

Another special class of these many-body problems is characterized (via a well-known mechanism, see section 4.2.3 of [3]) by the remarkable property to yield generic motions which are not periodic but approach asymptotically limit cycles all having the same period; we plan to discuss dynamical systems featuring this phenomenology (in the context of the models identified herein, and also more generally) in a separate paper [8]. Another direction of future investigation suggested by the results presented herein is to identify and investigate other systems whose solvability is connected with exceptional polynomial subspaces of codimension higher than 2 [5, 6].

## Acknowledgments

## References

[1]  Bruschi M and Calogero F 2006 Goldfishing: a new solvable many-body problem *J. Math. Phys.* **47** 042901 1–35
[2]  Calogero F 2001 *Classical Many-Body Problems Amenable to Exact Treatments (Lecture Notes in Physics Monograph* vol 66*)* (Berlin: Springer)
[3]  Calogero F 2007 *Isochronous Systems* (200-page monograph) at press
[4]  Calogero F and Gómez-Ullate D 2007 Two novel classes of solvable many-body problems of goldfish type with constraints *J. Phys. A: Math. Theor.* **40** 1–19
[5]  Gómez-Ullate D, Kamran N and Milson R 2004 Supersymmetry and algebraic Darboux transformations *J. Phys. A: Math. Gen.* **37** 10065–78
[6]  Gómez-Ullate D, Kamran N and Milson R 2007 A generalized Bochner problem and related orthogonal polynomials (in preparation)
[7]  Gómez-Ullate D and Sommacal M 2005 Periods of the goldfish many-body problem *J. Nonlinear Math. Phys.* **12** (Suppl. 1) 351–62
[8]  Calogero F and Gomez-Ullate D 2007 Many-body problems whose generic solutions all tend to limit cycles with the same period (in preparation)